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Sensitive Response of the Quantum Entropies to Jaynes-Cummings Model in Presence of a Second Harmonic Generation

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The behavior of a modified Jaynes-Cummings Hamiltonian model (two-level atom in interaction with an electromagnetic field) in the presence of degenerate parametric amplification is introduced. We have examined two different cases, one when the field frequency ω is not equal to the splitting photon frequency ε for which the off-resonance case is considered. In the second case we have taken each frequencies to be equal ($\omega = \varepsilon$) and considered the system to be at exact resonance. The wave function for both cases is obtained and the result used to calculate the density matrix from which we manage to discuss the field entropy as well as the phase entropy. It is shown that the system is sensitive to any change in the splitting photon frequency ε.

KEY WORDS: JC model-parametric amplifier; entropy squeezing.

1. INTRODUCTION

It is well known that the field of quantum mechanics gives rise to the notion of entangled states as a resultant of the superposition principle as well as the structure of the Hilbert space. Entangled states are states of two or more systems correlated with each other, but without classical features. In fact quantum entanglement is one of the most striking features of quantum mechanics (Bell, 1965; Einstein *et al.*, 1935). The recent surge of interest and progress in quantum information theory allows one to take a more positive view of entanglement and regard it as an essential resource for many ingenious applications such as quantum dense coding (Bennett and Wiesner, 1992), quantum teleportation (Bennett *et al.*, 1994), and quantum cryptography (Ekert, 1991). More complex entanglement manipulations could be used for quantum error correction (Steane, 1996) or entanglement purification (Van Enk *et al.*, 1997).

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On the other hand one can see that the preparation of complex entangled states under well-controlled conditions is the subject of intense experimental activity (Greenberger *et al.*, 1990). The manipulation of these states that have nonclassical and nonlocal properties leads to a better understanding of basic quantum phenomena. The manipulation of controlled entangled states, protected from their environment, is experimentally challenging. More precisely entanglement between individual systems has been achieved so far in quantum optics systems e.g. in photon down-conversion processes (Pan *et al.*, 2000), with trapped ions (Sackett *et al.*, 2000) or in cavity quantum electrodynamics (QED) (Rauschenbeutel *et al.*, 2000).

The entanglement in the last case results from the interaction of a two-level atom with a cavity field mode. With circular Rydberg atoms and superconducting cavities, the coherent atom-field coupling overwhelms dissipation (Haroche and Raimond, 1994), where the basic interaction process is the vacuum Rabi oscillation (Brune *et al.*, 1996). This indicates that the cavity (QED) system is an almost ideal system to generate entangled states and to perform small scale quantum information processing (Raimond *et al.*, 2001). This stimulated and encouraged us to turn our attention to the Jaynes-Cummings model (JC) which represents such type of the interaction (Jaynes and Cummings, 1963; Rauschenbeutel *et al.*, 1999).

The main purpose of the present paper is to examine the (JC) model, however, in the presence of a second harmonic generation process by the same cavity field (namely degenerate parametric amplification) (Abdalla *et al.*, 2005). Specifically we examine two main quantum aspects of the suggested model: Quantum field entropy as well as phase entropy. The model we adopt here is given by

$$\frac{\hat{H}}{\hbar} = \omega\hat{a}^{\dagger}\hat{a} + \xi(t)\hat{a}^{\dagger 2} + \xi^{*}(t)\hat{a}^{2} + \frac{\omega_0}{2}\hat{\sigma}_z + \lambda(\hat{a}^{\dagger} + \hat{a})(\hat{\sigma}_- + \hat{\sigma}_+), \tag{1}$$

where $\hat{a}^{\dagger}$ and $\hat{a}$ are the creation and annihilation operators for the cavity mode such that $[\hat{a}, \hat{a}^{\dagger}] = 1$, and ω and ω_0 are the field and the atomic transition frequencies respectively, while λ is the coupling constant between the field and the atom. The operators $\sigma_+(\sigma_-)$, and σ_z are the usual raising (lowering) and inversion operators for the two-level atomic system, satisfying $[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm$ and $[\sigma_+, \sigma_-] = \sigma_z$. The time-dependent complex function $\xi(t)$ is a response of the second harmonic generation (degenerate parametric amplifier) and is given by

$$\xi(t) = \frac{ik}{2}\exp(-2i\varepsilon t) \tag{2}$$

where k is an arbitrary constant and ε is the frequency of the split photon. It should be noted that the existance of the second harmonic generation term in the Hamiltonian (1) may reflect appreciable fluctuations in the strength of the cavity field, which may arise in a number of ways, for instance from external signals, from noise operators or maybe from other internal reactive effects. In the meantime the simplest and most illuminating example of a change in the field intensity is the case of damping; for instance, the field simply leaks away through the walls of the cavity. This means that for a realization of this model the interaction time t_{int} (say) should be such that $t_{\text{int}} \ll \kappa^{-1}$, where κ is the rate of leakage of the field out of the cavity. Cavities with high enough $Q = \omega/\kappa$ factor can guarantee that this condition is fulfilled. Further the time of interaction should be much shorter than the time scale of spontaneous decay, but however long enough to allow for appreciable exchange of energy between the atom and the field and between the field modes, i.e $\lambda^{-1}, k^{-1} \ll t_{\text{int}} \ll \gamma^{-1}, \kappa^{-1}$ where γ is the spontaneous decay rate. For more discussion see Puri (2001).

The organization of the paper is as follows: In Section II we derive the wave equation from which we are able to obtain the density matrix, however, for two different cases. The first case when the frequency of the split photon is not equal to the field frequency, while the second case is for when both frequencies are equal. Section III is devoted to the quantum field entropy and is followed by Section IV where we consider the phase entropy. Finally our conclusion is given in section V.

2. THE DENSITY MATRIX

To study the dynamics of the system we have to obtain the exact expression of the time-dependent wave function in the schrödinger picture. In this context we consider the case in which the split photon frequency ε is equal to the field frequency ω. However, we shall start with the case in which $\omega \neq \varepsilon$.

Case I. $(\omega \neq \varepsilon)$

To deal with this case we introduce the scaled time-dependent operators

$$\hat{A} = \hat{a}\exp(i\varepsilon t), \qquad \text{and} \qquad \hat{A}^{\dagger} = \hat{a}^{\dagger}\exp(-i\varepsilon t). \tag{3}$$

Thus, if we substitute Eq. (3) into the Hamiltonian (1), taking care with the generating function, then we have

$$\frac{\hat{H}}{\hbar} = \left[\delta\hat{A}^{\dagger}\hat{A} + i\frac{k}{2}(\hat{A}^{\dagger 2} - \hat{A}^{2})\right] + \frac{\omega_0}{2}\hat{\sigma}_z + \lambda(\hat{A}\exp(-i\varepsilon t) + \hat{A}^{\dagger}\exp(i\varepsilon t))(\hat{\sigma}_- + \hat{\sigma}_+). \tag{4}$$

where $\delta = (\omega - \varepsilon)$ and $\omega \neq \varepsilon$. Moreover, if we invoke the canonical transformation (Abdalla *et al.*, 2005)

$$\hat{A}^{\dagger} = \hat{B}^{\dagger}\cosh\phi + i\hat{B}\sinh\phi, \qquad \hat{A} = \hat{B}\cosh\phi - i\hat{B}^{\dagger}\sinh\phi, \tag{5}$$

where $\hat{B}$ and $\hat{B}^\dagger$ satisfy the commutator $[\hat{B}, \hat{B}^\dagger] = 1$, then Eq. (4) becomes

$$\frac{\hat{H}}{\hbar} = \Omega \hat{B}^\dagger \hat{B} + \frac{\omega_0}{2}\hat{\sigma}_z + \lambda[(\hat{B}\cosh\phi - i\hat{B}^\dagger \sinh\phi)\exp(-i\varepsilon t) + (\hat{B}^\dagger\cosh\phi + i\hat{B}\sinh\phi)\exp(i\varepsilon t)](\hat{\sigma}_- + \hat{\sigma}_+), \tag{6}$$

where

$$\phi = \frac{1}{2}\tanh^{-1}(k/\delta), \qquad \Omega = \sqrt{\delta^2 - k^2}. \tag{7}$$

In the interaction picture the Hamiltonian (6) is given by

$$\frac{V^I(t)}{\hbar} = \lambda\{\hat{B}(e^{-i\varepsilon t}\cosh\phi + i\sinh\phi e^{i\varepsilon t})e^{-i\Omega t} + \hat{B}^\dagger(e^{i\varepsilon t}\cosh\phi - i\sinh\phi e^{-i\varepsilon t})e^{i\Omega t}\}(\hat{\sigma}_- e^{-i\omega_0 t} + \hat{\sigma}_+ e^{i\omega_0 t}). \tag{8}$$

Now, if we apply the rotating wave approximation (RWA), for which we neglect the energy nonconserving terms $\hat{B}\hat{\sigma}_-$ and $\hat{B}^\dagger\hat{\sigma}_+$, then Eq. (8) reduces to

$$V^I(t) = \hbar\lambda[\hat{B}^\dagger\hat{\sigma}_- J(t) + \hat{B}\hat{\sigma}_+ J^*(t)], \tag{9}$$

where

$$J(t)\exp[-i\Omega t] = \cosh\phi\exp[-i(\omega_0 - \varepsilon)t] - i\sinh\phi\exp[-i(\omega_0 + \varepsilon)t]. \tag{10}$$

It should be noted that to avoid any appearance of nonconservative terms we have applied the RWA to the rotated operators $\hat{B}(\hat{B}^\dagger)$ not to the physical operators $\hat{A}(\hat{A}^\dagger)$. Further the rapidly oscillating terms, $\exp[\pm i(\omega_0 + \varepsilon)t]$, are neglected within the RWA and then the interaction Hamiltonian (9) takes the form,

$$\frac{V^I(t)}{\hbar} = \lambda\cosh\phi\{\hat{B}^\dagger\hat{\sigma}_- \exp[-i(\omega_0 - \varepsilon)t]\exp[i\Omega t] + \hat{B}\hat{\sigma}_+ \exp[i(\omega_0 - \varepsilon)t]\exp[-i\Omega t]\}. \tag{11}$$

Note that the transformed Hamiltonian (11) is a JC form of interaction attained by RWA, but with the canonically transformed field operators $\hat{B}$, $\hat{B}^\dagger$.

The continuous map $\mathcal{E}_t^*$ describing the time evolution between the atom and the field is defined by the unitary operator generated by $\hat{H}$ such that

$$\begin{aligned} \mathcal{E}_t^* &: S_A \longrightarrow S_A \otimes S_F, \\ \mathcal{E}_t^* \rho &= \hat{U}_t(\rho_A(0) \otimes \rho_F(0))\hat{U}_t^*, \\ \hat{U}_t &\equiv \exp\left(-\frac{1}{\hbar}\int_0^t \hat{H}(\grave{t})d\grave{t}\right). \end{aligned} \tag{12}$$

Bearing these facts in mind we find that the evolution operator $\hat{U}_t$ takes the following from

$$\hat{U}_t = \begin{bmatrix} \widehat{U}_{ee} & \widehat{U}_{eg} \\ \widehat{U}_{ge} & \widehat{U}_{gg} \end{bmatrix}, \tag{13}$$

where $\widehat{U}_{ij}$ is the single element matrix in the atomic subsystem basis, where $|e\rangle$ and $(|g\rangle)$ are the excited and the ground states of the atom respectively has the expressions

$$\begin{aligned}
\widehat{U}_{ee} &= \exp\left[-\frac{i(\Delta-\Omega)t}{2}\right]\left(\cos g_n t + i\frac{(\Delta-\Omega)}{2g_n}\sin g_n t\right), \\
\widehat{U}_{eg} &= -i\widetilde{g}\exp\left[-\frac{i(\Delta-\Omega)t}{2}\right]\left(\sqrt{n+1}\frac{\sin g_n t}{g_n}\right), \\
\widehat{U}_{gg} &= \exp\left[-\frac{i(\Delta-\Omega)t}{2}\right]\left(\cos g_{n-1} t + i\frac{(\Delta-\Omega)}{2g_{n-1}}\sin g_{n-1} t\right),
\end{aligned} \tag{14}$$

and $\widehat{U}_{ge}$ is its Hermitian conjugate. Finally the factor g_n is the Rabi frequency given by

$$g_n = \sqrt{\widetilde{g}^2(n+1) + \frac{1}{4}(\Delta-\Omega)^2}, \quad \widetilde{g} = \lambda\cosh\phi, \quad \Delta = \omega_0 - \epsilon. \tag{15}$$

It must be born in mind that the photon number n appearing in g_n must be treated as an operator. It should be noted that the second harmonic generation in this case has two effects: changing the Rabi vacuum frequency by a factor $\cosh\phi$ and adding to the detuning parameter.

Case II. $(\omega = \varepsilon)$

In order to consider the case in which $\omega = \varepsilon$, one may think of the limiting case. However, this is not the proper way to do so. This can be noted from the inconsistency which would appear as a consequence of the singularity in Eq. (7). Now, if we $\delta = 0$, we can rewrite the Hamiltonian (1) in the form

$$\frac{\hat{H}}{\hbar} = \omega\hat{a}^\dagger\hat{a} + \xi(t)\hat{a}^{\dagger 2} + \xi^*(t)\hat{a}^2 + \frac{\omega_0}{2}\hat{\sigma}_z + \lambda(\hat{a}^\dagger + \hat{a})(\hat{a}_- + \hat{\sigma}_+), \tag{16}$$

where

$$\xi(t) = \frac{ik}{2}\exp(-2i\omega t). \tag{17}$$

We follow the same procedure as before and define new time-dependent operators $\hat{A}_1$ and $\hat{A}_1^\dagger$ such that

$$\hat{A}_1 = \hat{a}\exp(i\omega t) \qquad \text{and} \qquad \hat{A}_1^\dagger = \hat{a}^\dagger \exp(-i\omega t) \tag{18}$$

Then Eq. (16) takes the form

$$\frac{\hat{H}}{\hbar} = \frac{\omega_0}{2}\hat{\sigma}_z + \lambda\big(\hat{A}_1\hat{\sigma}_+ \exp(-i\omega t) + \hat{A}_1^\dagger\hat{\sigma}_- \exp(i\omega t)\big) + i\frac{k}{2}\big(\hat{A}_1^{\dagger 2} + \hat{A}_1^2\big). \tag{19}$$

In terms of the interaction picture we have

$$\begin{aligned}\frac{V^I(t)}{\hbar} &= \lambda\{[\hat{A}_1\cosh kt + \hat{A}_1^\dagger \sinh kt]\hat{\sigma}_+ \exp(-i\Delta t)\\ &\qquad + [\hat{A}_1^\dagger \cosh kt + \hat{A}_1 \sinh kt]\hat{\sigma}_- \exp(i\Delta t)\},\end{aligned} \tag{20}$$

where $\Delta = (\omega_0 - \omega)$ is the detuning parameter. Here we point out that as a result of the degenerate parametric amplifier terms the interaction Hamiltonian acquires rapid oscillating terms which we ignore within the RWA. Hence Eq. (20) reduces to

$$\frac{V^I(t)}{\hbar} = \lambda\cosh kt\big(\hat{A}_1\hat{\sigma}_+ \exp(-i\Delta t) + \hat{A}_1^\dagger\hat{\sigma}_- \exp(i\Delta t)\big), \tag{21}$$

Now using Eqs. (12), (13) and Eq. (21), we get when Δ is zero:

$$\hat{U}_t = \begin{bmatrix}\widehat{U}_{ee} & \widehat{U}_{eg}\\ \widehat{U}_{ge} & \widehat{U}_{gg}\end{bmatrix}, \tag{22}$$

where the $\hat{U}_{ij}$ are given by

$$\begin{aligned}\widehat{U}_{ee} &= I_0(\bar{g}) + 2\sum_{r=1}^{\infty}(-)^r I_{2r}(\bar{g})\cosh(2rkt),\\ \widehat{U}_{eg} &= -2i\left(\sum_{r=0}^{\infty}(-)^r I_{(2r+1)}(\bar{g})\sinh[(2r+1]kt)\right),\\ \widehat{U}_{gg} &= \left(I_0(\bar{g}) + 2\sum_{r=1}^{\infty}(-)^r I_{2r}(\bar{g})\cosh(2rkt)\right),\end{aligned} \tag{23}$$

where $\bar{g} = (\lambda/k)\times\sqrt{n+1}$ is the modified Rabi frequency and $I_n(\cdot)$ is the modified Bessel function of order n. We may note here the formal difference between the two expressions in (14) and (23). The dependence on the parameter k appears in the argument of the modified Bessels functions as well as the time dependent hyperbolic functions. In the following sections we use the density matrix obtained here to discuss the effect of the second harmonic generation term on the JC model and to see that change would occur in both of field and phase entropies.

3. QUANTUM FIELD ENTROPY

As we have previously mentioned, quantum entanglement has become a subject for intensive study among those interested in the foundations of quantum theory, see for example (Nielsen and Chuang, 2000). Since the quantum dynamics described by the Hamiltonian (1) leads to an entanglement between the field and the atom, therefore in the next section of the present paper we use the field entropy as a measure for the degree of entanglement between the fields and the atom of the system under consideration. To achieve this goal we firstly define the system entropy as

$$\hat{S} = -\mathrm{Tr}\{\hat{\rho} \ln \hat{\rho}\}, \tag{24}$$

where $\hat{\rho}$ is the density operator for a given quantum system. It should be noted that Boltzmann's constant, K has been taken equal to unity. If the density matrix $\hat{\rho}$ describes a pure state, then $\hat{S} = 0$. However, if $\hat{\rho}$ describes a mixed state, then $\hat{S} \neq 0$, but according to the Araki-Lieb Theorem (Araki and Lieb, 1970) a system (the entropy of which is $\hat{S}$) consisting of two subsystem (the entropies of which are $\hat{S}_A$ and $\hat{S}_F$), these entropies satisfy the inequalities $|\hat{S}_A - \hat{S}_F| \leq \hat{S} \leq \hat{S}_A + \hat{S}_F$. Now suppose the field and the atom are treated as a separate system. Then the entropy can be defined through the corresponding reduced density operators by

$$\hat{S}_{A(F)} = -\mathrm{Tr}_{A(F)}\{\hat{\rho}_{A(F)} \ln \hat{\rho}_{A(F)}\}, \tag{25}$$

provided we treat both atom and field separately. The density matrix $\hat{\rho}_{A(F)}(t)$ in the above equation is given by

$$\hat{\rho}_{A(F)}(t) = \mathrm{Tr}_{F(A)}|\psi(t)\rangle\langle\psi(t)|, \tag{26}$$

where we have used the subscript $A(F)$ to denote the atom (field) respectively. We should note here that, if the atom-field system is initially in a pure state, then at any time $t > 0$, the entropies of the field and the atomic subsystems are precisely equal (Araki and Lieb, 1970; Phoenix and Knight, 1988, 1991a,b).

To this end if we consider that the initial state of the atom takes the following form

$$\hat{\rho}_A(0) = |\psi_A(0)\rangle\langle\psi_A(0)|, \tag{27}$$

where

$$|\psi_A(0)\rangle = [\cos\theta|e\rangle + e^{i\phi}\sin\theta|g\rangle]. \tag{28}$$

Furthermore we suppose that the initial state of the field is given by $\hat{\rho}_F(0) = |\varpi\rangle\langle\varpi|$, where $|\varpi\rangle = \sum_{n=0}^{\infty}, p_n|n\rangle$ and $p_n^2 = |\langle\varpi|n\rangle|^2$ being the probability distribution of photon number for the initial state. Thus from Eq. (13) the reduced field density operator takes the form

$$\hat{\rho}_F(t) = |C(t)\rangle\langle C(t)| + |S(t)\rangle\langle S(t)|, \tag{29}$$

where

$$|C(t)| = \sum_{n=0}^{\infty} p_n(\hat{U}_{ee}\cos\theta + e^{i\phi}\hat{U}_{eg}\sin\theta)|n\rangle,$$

$$|S(t)| = \sum_{n}^{\infty} p_n(\hat{U}_{ge}\cos\theta + e^{i\phi}\hat{U}_{gg}\sin\theta)|n\rangle. \tag{30}$$

In order to derive a computational formalism of the field entropy, we must obtain the eigenstates and corresponding eigenvalues of the reduced field density operator. A general method has been developed to calculate the various field eigenstates in a simple way, see Phoenix and Knight (1988, 1991a,b). By using this method we obtain the eigenvalues and eigenstates of the reduced density operator,

$$\begin{aligned}\lambda_f^{\pm}(t) &= \langle C(t)|C(t)\rangle \pm \exp[\mp\varrho]|\langle C(t)|S(t)\rangle| \\ &= \langle S(t)|S(t)\rangle \pm \exp[\pm\varrho]|\langle C(t)|S(t)\rangle|, \\ |\psi_f^{\pm}(t)\rangle &= \frac{1}{\sqrt{2\lambda_f^{\pm}(t)\cosh(\varrho)}}\{\exp[(i\vartheta \\ &\quad \pm\varrho)/2]C(t)\rangle \pm \exp[-(i\vartheta \pm \varrho)/2]|S(t)\rangle\},\end{aligned} \tag{31}$$

where

$$\varrho = \sinh^{-1}\left(\frac{\langle C(t)|C(t)\rangle - \langle S(t)|S(t)\rangle}{2|\langle C(t)|S(t)\rangle|}\right), \tag{32}$$

and $\langle C(t)|S(t)\rangle = |\langle C(t)|S(t)\rangle|\exp(i\vartheta)$. We can express the field entropy $S_F(t)$ in terms of the eigenvalue $\lambda_f^{\pm}(t)$ of the reduced field density operator,

$$\hat{S}_F(t) = -\lambda_f^{+}(t)\ln(\lambda_f^{+}(t)) - \lambda_f^{-}(t)\ln(\lambda_f^{-}(t)). \tag{33}$$

In the case of a disentangled pure joint state $\hat{S}_F(t)$ is zero and for maximally entangled states it gives $\ln 2$. To see the effect of the second harmonic generation (degenerate parametric amplifier) on the JC model we have plotted the time evolution of the quantum field entropy against the scaled time λt for the two above mentioned cases. In Fig. (1) we have considered the case in which the field frequency ω does not match the split frequency ε for $\bar{n} = 20$, but with different values of the other parameters. For example in Fig. 1(a) we have taken $\delta = 0.7\lambda$, $k = 0.5\lambda$, and the detuning parameter $\Delta = 0$, while the atom is in its excited state $\theta = 0$. This means that $\tanh 2\phi = 0.7$ and $\omega = 0.5\lambda$. In this case we realize that the maximum value of entanglement is achieved after short period of time (~ 0.65) more precisely after onset of the interaction. This is followed by a period of time where we can see a decreasing in its value to reach its minimum (~ 0.05). However, the function starts again to increase showing collapses with

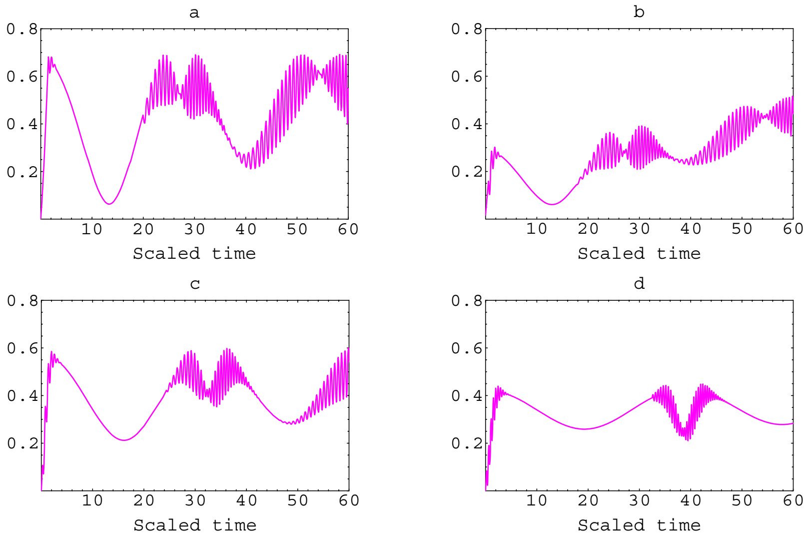

Fig. 1. The time evolution of the quantum field entropy as a function of the scaled time λt (Case I ($\omega \neq \varepsilon$)) for $\bar{n} = 20$, where (a) $\delta = 0.7\lambda$, $k = 0.5\lambda$, $\theta = 0$, and $\Delta = 0$, (b) As (a) but for intermediate atomic state, $\theta = \pi/3$ (c) As (a) but $\delta = 5\lambda$ and (d) As (a) but $\Delta = 10\lambda$.

rapid fluctuations, in the meantime we can observe a slight revival at the half period of the time considered. Another period of decreasing is observed, where the function reaches the value ~ 0.3, and then it starts to increase its value again with collapses and more rapid fluctuations. Thus the effect of the parameter k here amounts to a slight prolongation of time revival the standard JCM and slight lowering in the maximum value of entanglement due to the small detuning.

For the intermediate state case ($\theta = \pi/3$), we observe different behavior of the field entropy, as can be seen in Fig. 1(b). In this case after onset of the interaction the function decreases in value quite drastically compared with the excited state case. However as the time develops it starts again to increase in value (with rapid fluctuations) and to reach its maximum at the end of the time considered which is approximately ~ 0.5. As soon as we take $\delta = 5\lambda$ which means $\tanh 2\phi = 0.1$ and $\omega = 5\lambda$, the function in the excited state case increases its minimum to be just above 0.25, while the maximum entanglement is slightly reduced, see Fig. 1(c). This means that the interaction between the atom and the field gets stronger (in certain period of time) compared with the case in which $\delta = 0.7\lambda$. Consequently we can say that the system is somehow sensitive to the change in the frequency of the split photon.

Finally, when we take the effect of the detuning paremeter into account $\Delta = 10\lambda$, we can see that there is a decreases in the maximum value of the entropy with an increase in its minimum. Also there is a long period of revival which refers to strong correlation between the field and the atom, see Fig. 1(d).

To continue our progress we discuss the effect of the degenerate parametric amplifier on the JC model for the case in which $\omega = \varepsilon$. For this reason we have plotted as before in Fig. (2) the time evolution of the quantum field entropy against the scaled time λt. We have considered the system in the excited state ($\theta = 0$) and take $\bar{n} = 20$ and $\Phi = 0$, while $k = 0.1\lambda$. In this case the situation is quite different compared with the previous case. This is due to the sensitivity of the system to the variation that would occur in the the photon split frequency. For example, we can realize that the function increases in value to reach the maximum, however, at a time later than the case $\omega \neq \varepsilon$. This is followed by a period of decrease where the function reduces its value after a period of time almost twice the $\omega = \varepsilon$ case. Further we can see another period of time in which the function reaches its maximum with rapid fluctuations and interference between the pattern showing a long period of collapses, see Fig. 2(a). As soon as we increase the value of the parameter k, such that $k = 0.5\lambda$, the function reaches its maximum after onset of the interaction and then decreases in value to reach its minimum in period of time much less than half of the period in the non-degenerate case of Fig 1(a). This is followed by an increase in value again, where we can see a long period of rapid fluctuations and interference between the pattern. This means that the system is in strong interaction with maximum entanglement around (~ 0.7), see Fig. 2(b).

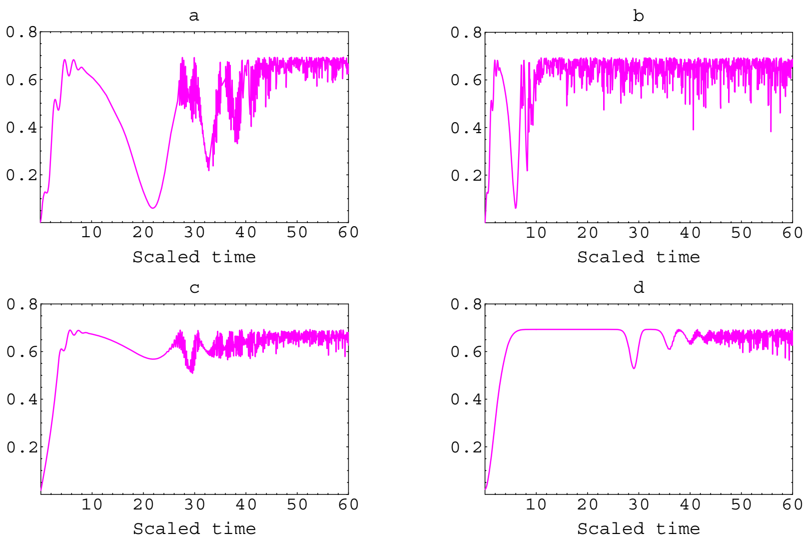

Fig. 2. As Fig. 1, but for Case II :($\omega = \varepsilon$), $\bar{n} = 20$, and for: (a) $\theta = \Phi = 0$, $k = 0.1\lambda$, (b) As (a) but for $k = 0.5\lambda$, (c) As (a) but for $\theta = \pi/3$, (d) As (a) but for $\theta = \pi/4$.

For the intermediate state case we have considered two cases: one at $\theta = \pi/3$, and the other at $\theta = \pi/4$. In both cases the field entropy takes its maximum immediately after onset of the interaction, and the general behavior is almost the same. However, the period of collapse is different for we can see that the period of revival for the case $\theta = \pi/4$ is longer than the case of $\theta = \pi/3$. Moreover the function at $\theta = \pi/3$ has more fluctuations than the other case, see Figs. 2(c,d). It is to be mentioned that strong entanglement persists after the onset of interaction for longer periods in contrast to the case of the non-degenerate interaction considered earlier. This may be compared with the results given by Phoenix and Knight (1988, 1991a,b), where this phenomenon is absent from their model since they considered the JCM with one and two photon processes. Furthermore for the case in which $\omega \neq \varepsilon$ we observe that there is a reduction in the value of the maximum entanglement (for the same value of parameters, as well the time of consideration) compared to the earlier case considered in this reference. Thus we can conclude that the addition of the second harmonic generation certainly leads to appreciable modifications in the degree of entanglement.

4. PHASE ENTROPY

The main purpose of this section is to discuss the effect of the second harmonic generation on the JC model through the behavior of the phase entropy. For the phase description there are different techniques to deal with it. Some of them are based on a Hermitian quantum phase operator or associated with quasiprobability distribution functions in a phase space. However, one can find that the approach which is based on operational definition of quantum phase is more convenient. Therefore in the present paper we use the Shannon entropy associated with the phase probability distribution P_α, which is given by

$$P_\alpha = \langle\alpha|\hat{\rho}(t)|\alpha\rangle, \tag{34}$$

where $|\alpha\rangle$ is the phase state and $\hat{\rho}$ is the density matrix. The Shannon entropy for the density operator given by the above equation is (Bialynicki-Birula and Mycielski, 1975; Deutsch, 1983; Maassen and Uffink, 1988)

$$R_\psi = -\int_{2\pi} (P_\alpha \ln P_\alpha)\, d\alpha. \tag{35}$$

The single-mode of the Pegg-Barnett phase formalism which of interest in the field of quantum optics can be constructed from the single-mode phases (Obada *et al.*, 1998; Pegg and Barnett, 1989) to take the form

$$P(\alpha, t) = \lim_{s\to\infty} \left(\frac{s+1}{2\pi}\right) \langle\alpha_m|\hat{\rho}(t)|\alpha_m\rangle, \tag{36}$$

where $|\alpha_m\rangle$ is a phase state of the mode and given by

$$|\alpha_m\rangle = \frac{1}{\sqrt{(s+1)}} \sum_{n=0}^{s} e^{in\alpha_m}|n\rangle, \tag{37}$$

and $\alpha_m = \alpha_\circ + \frac{2\pi m}{s+1}$ with $m = 0, 1, \ldots, s$, and $\alpha_\circ$ arbitrary. Eq. (37) defines a particular basis set of $(s+1)$ mutually orthogonal phase states. Using the standard procedure, the phase probability distribution, and the expectation value as well as the variance of the Hermitian phase operator may be obtained for the field. Since the coherent field at $t = 0$ belongs to a class of partial phase states, therefore we have chosen the reference phase α_0 as $\alpha_0 = \beta - \frac{\pi s}{s+1}$ and introduced the new phase labels $\zeta = m - \frac{1}{2}s$ where $m = 0, 1, 2, \ldots, s$. Then, as s tends to infinity, the summation may be transformed into an integral after replacing $\frac{2\pi\zeta}{s+1}$ by α and $\frac{2\pi}{s+1}$ by $d\alpha$. This leads to continuous phase probability distribution, where

$$P(\alpha, t) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (\hat{\rho}(t))_{nm} \exp[i\alpha(n-m)] \ , \tag{38}$$

which is normalized according to

$$\int_{-\pi}^{\pi} P(\alpha, t)\, d\alpha = 1. \tag{39}$$

Now we discuss the behavior of the Pegg-Barnett phase for the present system. For this reason we have plotted Figs. 3(a) and (3b). The first figure describes the case in which the split frequency ε does not equal the field frequency ω, where we have taken into consideration the system in the excited state and the mean photon number $\bar{n} = 20$. In the meantime we have kept the value of the other parameters

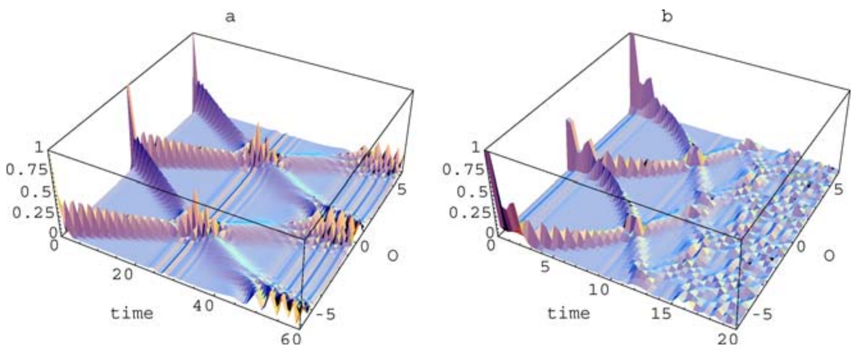

Fig. 3. $P(\alpha, t)$ against α and the scaled time λt. The mean photon number is fixed for all cases, such that $\bar{n} = 20$. (a) case I ($\omega \neq \varepsilon$) with the same parameters as in Fig. (1a) and (b) case II ($\omega = \varepsilon$) with the same parameters as in Fig. 2(a).

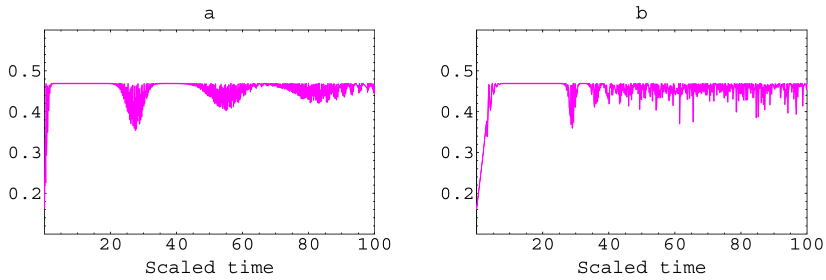

Fig. 4. Phase entropy as a function of the scaled time λt. The mean photon number is fixed for all cases, such that $\bar{n} = 20$. (a) case I $\omega \neq \varepsilon$ with the same parameters as in Fig. 1a and (b) case II ($\omega = \varepsilon$) with the same parameters as in Fig. 2(a).

of Fig. 1(a) unchanged. If we consider the variation of θ as $-\pi \leq \theta \leq \pi$, in this case we realize at $t = 0$ (corresponding to the initial coherent state) the phase distribution, $P(\alpha, t)$, starts with a single-peak structure at $\theta = 0$ and as the time develops, the peaks splits into two diverging peaks until they reach the boundary at $\pm\pi$. As time increases the two peaks move onward converging until they meet at $\theta = 0$. Increasing the time by a further amount leads to breaking of *the peak structure to multipeak* and diffusion. For the second case for which $\omega = \varepsilon$, we have taken the same values as before of the parameters considered. In this case we realize that although the single peak structure persists for some time before it breaks into two diverging peaks. These two peaks meet the borders $\pm\pi$ at the half of the revival time, but after that they converge again with an increased velocity to the point $\theta = 0$. After that the phase diffuses and multipeaks are observed.

The phase entropy depicted in Fig. 4(a) for the non-degenerate case shows collapses and revivals of the entanglement with fluctuations around the revival time and a higher degree of entanglement between the times of revivals. However, for the degenerate case the phase entropy shows an increase in its value to its maximum after a short time from the onset of the interaction. Then a time of persistence of the stronger entanglement is followed by fluctuations. It is to be remarked that the phase entropy behavior exhibits the main features exposed by the phase distribution, where fluctuations in the entropy occur during the diffusion of the phase as can be seen from Figs. (3) and (4).

5. CONCLUSION

In this paper we have introduced a modified Jaynes-Cummings model to include the effect of the second order harmonic generation (degenerate parametric amplifier). We have handled two different cases for the present model: The first when the field frequency ω is not equal to the split photon frequency ε, where the canonical transformation is invoked to obtain the wave function in the Schrödinger picture for the off-resonance case. In the second case considered both frequencies are equal and therefore the wave function is obtained at exact resonance. The density matrix elements have been calculated by using the exact expression of the wave function for each case separately. The degree of entanglement is discussed through the field entropy. Also we have considered the phase entropy. We have shown that both of field and phase entropies are sensitive for any change in the splitting photon frequency ε.

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